

## Solutions 5

### Exercise 10.1

$$\begin{aligned}\hat{\theta} &= \mathbb{E}(\theta|\mathbf{x}) = \int \theta p(\theta|\mathbf{x}) d\theta \\ &= \int \theta \frac{p(\mathbf{x}|\theta)p(\theta)}{p(\mathbf{x})} d\theta \\ &= \frac{\int \theta p(\mathbf{x}|\theta)p(\theta) d\theta}{\int p(\mathbf{x}|\theta)p(\theta) d\theta} \\ &= \frac{\int \theta p(\mathbf{x}|\theta)\delta(\theta - \theta_0) d\theta}{\int p(\mathbf{x}|\theta)\delta(\theta - \theta_0) d\theta} \\ &= \theta_0 \frac{p(\mathbf{x}|\theta_0)}{p(\mathbf{x}|\theta_0)} \\ &= \theta_0\end{aligned}$$

The MMSE estimator is just the true value since our prior knowledge is perfect.

### Exercise 10.2

$$\begin{aligned}p_x(x[0], x[1]) &= p_w(x[0] - A, x[1] - A|A) \\ &= p_w(x[0] - A)p_w(x[1] - A) \\ &= p_x(x[0]|A)p_x(x[1]|A)\end{aligned}$$

Thus,  $x[0]$  and  $x[1]$  are conditionally independent. However, the covariance of  $x[0]$  and  $x[1]$  is

$$\begin{aligned}\text{Cov}(x[0], x[1]) &= \mathbb{E}[(x[0] - \mathbb{E}[x[0]])(x[1] - \mathbb{E}[x[1]])] \\ &= \mathbb{E}[(A + w[0])(A + w[1])] \\ &= \mathbb{E}[A^2] + \mathbb{E}[A] \mathbb{E}[w[0]] + \mathbb{E}[A] \mathbb{E}[w[1]] + \mathbb{E}[w[0]] \mathbb{E}[w[1]] \\ &= 1\end{aligned}$$

Thus,  $x[0]$  and  $x[1]$  are correlated, which means they are not independent unconditionally.

### Exercise 10.3

$$p(\mathbf{x}|\theta) = e^{-\sum_n (x[n] - \theta)} u(\min x[n] - \theta)$$

$$\begin{aligned}
p(\theta|\mathbf{x}) &= \frac{p(\mathbf{x}|\theta)p(\theta)}{\int p(\mathbf{x}|\theta)p(\theta)} d\theta \\
&= \frac{e^{-N(\bar{x}-\theta)e^{-\theta}} u(\min x[n] - \theta) u(\theta)}{\int_0^{\min x[n]} e^{-N(\bar{x}-\theta)} e^{-\theta} d\theta} \\
&= \frac{(N-1)e^{\theta(N-1)}}{e^{(N-1)\min x[n]} - 1} u(\min x[n] - \theta) u(\theta) \\
\hat{\theta} = \mathbb{E}(\theta|\mathbf{x}) &= \frac{\int_0^{\min x[n]} \theta(N-1)e^{\theta(N-1)} d\theta}{e^{(N-1)\min x[n]} - 1} \\
&= \frac{N-1}{e^{(N-1)\min x[n]} - 1} \int_0^{\min x[n]} \theta e^{\theta(N-1)} d\theta \\
&= \frac{\min x[n] e^{(N-1)\min x[n]} - 1}{e^{(N-1)\min x[n]} - 1} - \frac{1}{N-1}
\end{aligned}$$

So,

$$\hat{\theta} = \frac{\min x[n] \cdot e^{(N-1)\min x[n]} - 1}{e^{(N-1)\min x[n]} - 1} - \frac{1}{N-1}$$

#### Exercise 10.4

$$\begin{aligned}
p(\mathbf{x}|\theta) &= \frac{1}{\theta^N} u(\min x[n]) u(\theta - \max x[n]) \\
p(\theta) &= \frac{1}{\beta} u(\theta) u(\beta - \theta) \\
p(\theta|\mathbf{x}) &= \frac{(N-1)}{\theta^N (\max x[n]^{-(N-1)} - \beta^{-(N-1)})} \\
\hat{\theta} &= \frac{N-1}{N-2} \cdot \frac{(\max x[n])^{-(N-2)} - \beta^{-(N-2)}}{(\max x[n])^{-(N-1)} - \beta^{-(N-1)}}
\end{aligned}$$

Note that for  $\beta$  and  $N$  large, we have

$$\hat{\theta} \approx \frac{N-1}{N-2} \max x[n] \approx \max x[n]$$

which agrees with the MLE.

#### Exercise 10.5

$$\begin{aligned}
\text{Bmse}(\hat{\theta}) &= \mathbb{E}_{x,\theta} \left\{ [(\theta - \mathbb{E}(\theta|\mathbf{x})) + (\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})]^2 \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x} \left\{ [(\theta - \mathbb{E}(\theta|\mathbf{x})) + (\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})]^2 \right\} \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x} \{(\theta - \mathbb{E}(\theta|\mathbf{x}))^2\} + 2\mathbb{E}_{\theta|x} \{(\theta - \mathbb{E}(\theta|\mathbf{x}))(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})\} + \mathbb{E}_{\theta|x} \{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\} \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x} \{(\theta - \mathbb{E}(\theta|\mathbf{x}))^2\} + \mathbb{E}_{\theta|x} \{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\} \right\}
\end{aligned}$$

To minimize  $\text{Bmse}(\hat{\theta})$ , is equal to minimize  $\mathbb{E}_{\theta|x} \{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\}$ . Thus,  $\hat{\theta} = \mathbb{E}(\theta|x)$

**Exercise 10.6**

$$p(x[n]|A) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_+^2}} e^{-\frac{1}{2\sigma_+^2}(x[n]-A)^2} & A \geq 0 \\ \frac{1}{\sqrt{2\pi\sigma_-^2}} e^{-\frac{1}{2\sigma_-^2}(x[n]-A)^2} & A < 0 \end{cases}$$

$$p(\mathbf{x}|A) = \begin{cases} \frac{1}{(2\pi\sigma_+^2)^{N/2}} e^{-\frac{1}{2\sigma_+^2} \sum_n (x[n]-A)^2} & A \geq 0 \\ \frac{1}{(2\pi\sigma_-^2)^{N/2}} e^{-\frac{1}{2\sigma_-^2} \sum_n (x[n]-A)^2} & A < 0 \end{cases}$$

For  $\sigma_+^2 = \sigma_-^2$ ,  $p(\mathbf{x}|A) = p(\mathbf{x}; A)$ , and for  $\sigma_+^2 \neq \sigma_-^2$ ,  $p(\mathbf{x}|A) \neq p(\mathbf{x}; A)$

**Exercise 10.10**

$$\begin{aligned} p(\theta|\mathbf{x}) &= \frac{p(\mathbf{x}|\theta)p(\theta)}{\int p(\mathbf{x}|\theta)p(\theta)d\theta} \\ &= \frac{\theta^N e^{-\theta \sum_{n=0}^{N-1} x[n]} u(\min x[n]) \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} u(\min \theta)}{\int_{\min \theta}^{\infty} \theta^N e^{-\theta \sum_{n=0}^{N-1} x[n]} \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} d\theta} \\ &= \frac{\theta^N e^{-\theta \sum_{n=0}^{N-1} x[n]} u(\min x[n]) \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} u(\min \theta)}{\frac{\Gamma(N+\alpha)}{(\sum_{n=0}^{N-1} x[n] + \lambda)^{(N+\alpha)}} \int_{\min \theta}^{\infty} \theta^{N+\alpha-1} e^{-\theta(\sum_{n=0}^{N-1} x[n] + \lambda)} \frac{(\sum_{n=0}^{N-1} x[n] + \lambda)^{(N+\alpha)}}{\Gamma(N+\alpha)} d\theta} \\ &= \frac{(\sum_{n=0}^{N-1} x[n] + \lambda)^{(N+\alpha)} \theta^{N+\alpha-1} e^{-\theta(\sum_{n=0}^{N-1} x[n] + \lambda)} u(\min x[n]) u(\min \theta)}{\Gamma(N + \alpha)} \end{aligned}$$

Therefore,  $p(\theta|\mathbf{x})$  has the same form with  $p(\theta)$ .

**Exercise 10.11**

From Fig. 10.9a, we can see the PDF of the random vector  $[h, w]^T$  is correlated within an ellipse. This could be a Gaussian PDF. Hence, we estimate  $w$  based on  $h$  using a MMSE estimation from (10.16)

$$\hat{w} = \mathbb{E}(w) + \frac{\text{Cov}(h, w)}{\text{var}(h)}(h - \mathbb{E}(h))$$

From Fig. 10.9b, there does not appear to be any correlation between height and weight. Thus,  $\hat{w} = \mathbb{E}(w) = 150$  for all height.